Tutorial 7

Chapter 15 Practice Exercises

37. (a) Solve $x^2+y^2+z^2=8$ and $z=2$, we obtain $x^2+y^2=r^2=4$. Thus the volume is

$$V=\int_0^{2\pi}\int_0^2\int_2^{\sqrt{8-r^2}} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^2 (r\sqrt{8-r^2}-2r)\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[-\frac{1}{3}(8-r^2)^{3/2}-r^2\right]_0^2 d\theta=\int_0^{2\pi}\left(-\frac{8}{3}-4+\frac{1}{3}8^{\frac{3}{2}}\right)d\theta=\frac{4\sqrt{2}-5}{3}\cdot 8\pi$$

(b) $z\geq 2 \Rightarrow \rho\cos\phi\geq 2 \Rightarrow \rho\geq 2\sec\phi$; $x^2+y^2+z^2=\rho^2\leq 8 \Rightarrow \rho\leq 2\sqrt{2}$; $x^2+y^2=8-z^2\leq 4$
$\Rightarrow \rho^2\sin^2\phi\leq 4 \Rightarrow \sin^2\phi\leq 4/\rho^2\leq\cos^2\phi \Rightarrow 0\leq\tan\phi\leq 1 \Rightarrow 0\leq\phi\leq\frac{\pi}{4}$. Thus,

$$V=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{2\sec\phi}^{\sqrt{8}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{8}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}(2\sqrt{2}\sin\phi-\sec^3\phi\sin\phi)\,d\phi\,d\theta$$

$$=\frac{8}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}(2\sqrt{2}\sin\phi-\sec^2\phi\tan\phi)\,d\phi\,d\theta=\frac{8}{3}\int_0^{2\pi}\left[-2\sqrt{2}\cos\phi-\frac{1}{2}\tan^2\phi\right]_0^{\frac{\pi}{4}}d\theta$$

$$=\frac{8}{3}\int_0^{2\pi}\left(-2+2\sqrt{2}-\frac{1}{2}\right)d\theta=\frac{16}{3}\pi\left(-\frac{5}{2}+2\sqrt{2}\right)=\frac{4\sqrt{2}-5}{3}\cdot 8\pi$$

Chapter 15 Additional and Advanced Exercises.

4. Let $x^2+y^2+(x^2+y^2)^2=2$, we obtain $x^2+y^2=1$. Thus, the volume is

$$V=\int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} dz\,r\,dr\,d\theta=\int_0^{2\pi}\int_0^1\left(\sqrt{2-r^2}-r^2\right)\frac{1}{2}\,dr^2\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\left[-\frac{2}{3}(2-r^2)^{\frac{3}{2}}-\frac{1}{2}r^4\right]_0^1 d\theta=\frac{1}{2}\int_0^{2\pi}\left(-\frac{2}{3}-\frac{1}{2}+\frac{2}{3}\cdot 2^{\frac{3}{2}}\right)d\theta=\frac{8\sqrt{2}-7}{6}\pi$$

8. Since $r=3\sin\theta\geq 0 \Rightarrow \theta\in[0,\pi]$. $\forall\theta\in[0,\pi]$, $(3\sin\theta)^2\leq 9$. Thus, the volume is

$$V=\int_0^{\pi}\int_0^{3\sin\theta}\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta=2\int_0^{\pi}\int_0^{3\sin\theta}\sqrt{9-r^2}\,r\,dr\,d\theta=\int_0^{\pi}\int_0^{3\sin\theta}\sqrt{9-r^2}\,dr^2\,d\theta$$

$$=-\frac{2}{3}\int_0^{\pi}(9-r^2)^{\frac{3}{2}}\Big|_0^{3\sin\theta}d\theta=-\frac{2}{3}\int_0^{\pi}9^{\frac{3}{2}}\left[(\cos^2\theta)^{\frac{3}{2}}-1\right]d\theta=18\int_0^{\pi}1-|\cos\theta|^3\,d\theta$$

$$=18\left(\pi-\int_0^{\frac{\pi}{2}}\cos^3\theta\,d\theta+\int_{\frac{\pi}{2}}^{\pi}\cos^3\theta\,d\theta\right)=18\left(\pi-2\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\sin\theta\right)$$

$$=18\left(\pi-2\int_0^{\frac{\pi}{2}}(1-\sin^2\theta)\,d\sin\theta\right)=18\left(\pi-2\left[\sin\theta-\frac{1}{3}\sin^3\theta\right]_0^{\frac{\pi}{2}}\right)$$

$$=18\left(\pi-\frac{4}{3}\right)$$